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ONE DIMENSIONAL NONLINEAR WAKE-FIELDS EXCITED IN A COLD PLASMA BY CHARGED BUNCHES

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Abstract

One dimensional nonlinear plasma wake-fields excited by a single bunch and by series of bunches are considered. Essential differences are brought to light between negatively charged bunch case and positively one. The bunches with nonuniform distributions of density are investigated. The obtained results shows dependence of excited potential electric fields on bunches parameters and allows to choose these parameters optimal.

1 Introduction

The electromagnetic waves, excited in plasma by charged bunches, can be used both for focusing of bunches and for charge acceleration [1]. In the relativistic bunch case an amplitude of excited one dimensional nonlinear plasma waves can essentially exceed the conventional wavebreaking field $E_* = m_e v_0 \omega_p / e$ (v_0 is the phase velocity, $\omega_p = (4\pi n_0 e^2 / m_e)^{1/2}$ is the electron plasma frequency, n_0 - plasma electron density in equilibrium) and reaches of the value $\tilde{E} = [2(\gamma - 1)]^{1/2}$, where $\gamma = (1 - \beta^2)^{-1/2}$, $\beta = v_0/c$, and \tilde{E} is normalized on E_* . Acceleration rate in the wake-fields can reaches of a few GeV/m, that is much more, than was reached on conventional accelerators.

The one dimensional nonlinear wake-fields theory was developed in series of works (see e.g. [2]-[8] and references therein). Maximum of the accelerated electrons energy, as it was shown in [8], can reach of value $4m_e c^2 \gamma^3$.

In this paper bunches with the different parameters investigated analytically and by numerical simulation (including nonuniform and positively charged bunches).

2 One Dimensional Nonlinear Wake-fields

Consider a cold electron plasma with moving through the plasma, in Z direction charged bunch (or train of bunches), which transverse sizes one can consider infinite. The subject of our investigation is the steady wakefields excited by such bunches. Plasma ions assumed immobile because of their large mass ($m_e/m_i \ll 1$). As usually we also assume, that field in front of the bunch (or bunches) is absent, potential and strength of the electric field are continuous in all space. Then, from the continuity equation, relativistic equation of motion for plasma electrons and the Poisson equation one can obtain following equation for steady fields [5],[8]

$$\frac{d^2 F}{dz^2} + \beta^2 \gamma^2 \left(1 - \frac{\beta F}{(F^2 - \gamma^{-2})^{1/2}} \right) + \beta^2 \alpha(z) = 0, \quad (1)$$

where $F = 1 + |e|\varphi/m_e c^2 \geq \gamma^{-1}$, φ is the electric potential, $z = k_p(Z - v_0 t)$, $k_p = \omega_p/v_0$ (here the phase velocity v_0 is equal to bunch velocity), $\alpha(z) = (q/|e|)n_b(z)/n_0$, e is the

electron charge, q is the charge of the bunch particles, $n_b(z)$ is their density. Normalized on E_* the strength of electric field obeys the formula $E_z(z) \equiv E = -(1/\beta^2)dF/dz$. The plasma electrons dimensionless velocity as function of F is

$$\beta_e = [\beta - (F^2 - \gamma^{-2})^{1/2}]/(\beta^2 + F^2). \quad (2)$$

Inside uniform bunches and outside of bunches Eq. (1) can be rewritten in the form [5]

$$\begin{aligned} \frac{d^2 F}{dz^2} + \frac{dU}{dF} &= 0, \\ U &= \beta^2[(\gamma^2 + \alpha)F - \beta\gamma^2(F^2 - \gamma^{-2})^{1/2}]. \end{aligned} \quad (3)$$

Formally Eq. (3) describes one-dimensional motion of a particle in a field with potential $U(F)$. Analysis of the function $U(F)$ allows to determine both qualitative behaviour of the field and some quantitative values (such as electric field amplitude in the bunch, the wave-breaking field, maximum value of electric potential and other). Moreover, it can be obtained analytical solution of Eq. (3), which includes the elliptic functions [3],[5]-[7]. However, even in the case of uniform bunch, the analytical description of the field behind bunch in general case is practically impossible. For nonuniform bunches the problem analytically is not solved yet.

Outside the bunch (where $\alpha(z) = 0$) "potential" $U(F)$ has one minimum, therefore the field is periodical and for fixed γ fully determines by electric field amplitude $E_{mp} \leq \tilde{E}$. It is necessary to note that Eq. (3) with $\alpha = 0$ describes also the wake wave, excited by other way (for example, by laser pulse).

Nonlinear wake wavelength Λ_p increases as the amplitude E_{mp} increases. When $\gamma \gg 1$ the dependence $\Lambda_p(E_{mp})$ practically not depend on γ and is presented on Figure 1 (notice, that according to accepted in (1) variables, value $\Lambda_p = 2\pi$ corresponds to the linear approach). Near by the "breaking" ($E_{mp} \approx \tilde{E}$) the wake wavelength in the case of $\gamma \gg 1$ is approximately equal to $4(2\gamma)^{1/2}$ [3], [6] and maximum value of the dimensionless electric potential is $F_{mp} \approx (1 + \beta^2)\gamma$ [5]. In general case F_{mp} increases as E_{mp} increases (see Fig.1).

The energy gain of electrons (or positrons) accelerating in the wake wave can reach of value $4m_e c^2 \gamma^3$ [5]. Really, the relativistic equation of motion of accelerating electron

in the frame of reference moving with the wave velocity is

$$d(\beta'_a \gamma'_a)/dt' = cdF'/dZ'. \quad (4)$$

Taking into account, that $dZ' = c\beta'_a dt'$ and $\beta'_a d(\beta'_a \gamma'_a) = d\gamma'_a$ from (4) follows $\gamma'_a = \gamma'_a(0) + F' - F'(0)$, where $\gamma'_a(0)$ and $F'(0)$ are initial values. In the laboratory frame of reference γ_a one can obtain using known relativistic transformations $\varphi = \gamma\varphi'$ and $\gamma_a = \gamma\gamma'_a(1 + \beta\beta'_a)$. In the case $\gamma \gg 1$ and $\beta_a(0) \geq \beta$, choosing $F(0) \approx 1/\gamma$ and $F \approx F_{mp} \approx 2\gamma$ (that corresponds to $E_{mp} \approx \tilde{E}$) obtain $\gamma_a \approx 4\gamma^3$. In this case acceleration occur on the half of the wavelength; the acceleration length is proportional to $\gamma^{5/2}$ [5]. For example, when $n_0 = 3 \cdot 10^{12} \text{ cm}^{-3}$ and $\gamma = 10$, the acceleration length is $l_a \approx 1m$, $\Lambda_p \approx 5cm$ and $(\gamma_a)_{max} \approx 4000$. The result $(\gamma_a)_{max} \sim \gamma^3$ is consequence both of correlations $E_{mp} \sim \gamma^{1/2}$, $\Lambda_p \sim \gamma^{1/2}$ and relativistically large mass of the accelerating particle (i.e. $\beta_a \approx \beta \approx 1$), that causes long keeping back of the particle in accelerating phase. For arbitrary wake wave amplitude the maximum value F_{mp} can be obtained from Fig.1.

For the large γ the acceleration length can surpasses the laboratory plasma sizes. In the cosmos (for example, in the blanket of supernova), where the plasma can be considered unlimited, acceleration in the strong plasma fields described here can originate high energy cosmic rays.

3 Single Bunch

Consider a single uniform bunch: $n_b = \text{const}$, when $-d \leq z \leq 0$ and $n_b = 0$ elsewhere. The field inside the uniform bunch describes by Eq. (3). "Potential" $U(F)$ is maximum when $F = 1/\gamma$ and is minimum when

$$F = F_c = |\gamma^2 + \alpha|/(\gamma\Gamma^{1/2}), \Gamma = \gamma^2(1 + 2\alpha) + \alpha^2. \quad (5)$$

From (5) follows, that in the case $-1/(1+\beta) \equiv \alpha_1 < \alpha < \infty$ function $U(F)$ has minimum and the field inside bunch is periodical; F changes in the limits

$$1/\gamma \leq F \leq [(\gamma^2 + \alpha)^2 + \beta^2\gamma^4]/\gamma\Gamma. \quad (6)$$

When $\alpha = \alpha_1$, $U(F)$ monotonously decreases as F increases and tends to zero when $F \rightarrow \infty$. In this case the electric field strength inside the bunch grows from zero at the bunch head ($z = 0$) and tends to some constant value (see (7)). When $\alpha < \alpha_1$, $U(F)$ decreases unlimitedly, E grows from the bunch head to its tail ($z = -d$), and for bunch length d more than some value d_{max} the wake behind the bunch "breaks".

When $U(F(z = 0) = 1) > U_{max} = U(F = 1/\gamma)$ (that takes place, when $\alpha > \gamma$), in enough long bunch F becomes less than $1/\gamma$, that is using in this paper cold fluid approach becomes inapplicable. Thus, in the cases $-\infty < \alpha < \alpha_1$ and $\alpha > \gamma$ the uniform bunch length can not exceed some value d_{max} ; the wake amplitude in the case of $d = d_{max}$ is equal to \tilde{E} . Dependence of the d_{max} on α for $\gamma = 10$ presented on Figure 2.

In the case of $\alpha_1 \leq \alpha \leq \gamma$ the uniform bunch length can be arbitrary. The field inside the bunch is periodical and has amplitude [5], [6]

$$E_{mb} = [2(1 + \alpha - \Gamma^{1/2}/\gamma)]^{1/2}/\beta. \quad (7)$$

Dependence of the wavelength inside the bunch Λ_b on α for $\gamma \gg 1$ shown on Figure 3.

In the linear approach $E_{mp} \sim \alpha$ ($|\alpha| \ll 1$) and differences between the electron bunch case and the positron one reduce to changing of electric field sign (saying electron or positron bunch we mean negatively or positively charged bunch, respectively). In nonlinear theory these differences are essential. The wavelength Λ_b in the electron bunch increases as $|\alpha|$ increases when $\alpha_1 < \alpha < 0$ (see Fig.3), and in the case $\gamma \gg 1$ and $\alpha \approx -0.5$ is nearly equal to 8γ [3], [5], [6], that is much more than linear plasma wavelength. In the positron bunch Λ_b decreases as α increases. Note, that difference of the nonlinear wavelength from linear one, in the case $\gamma \gg 1$ is palpable even for $|\alpha| \ll 1$. From (3), (5) and the boundary conditions one can see, that $F(z) \geq 1$ in the electron bunch, and $F(z) \leq 1$ in the positron bunch. That is, according to (2), $\beta_e(z) \leq 0$ in the electron bunch and $\beta_e(z) \geq 0$ in the positron one.

Equation(1) was investigated numerically for uniform, parabolic and linear charge density distributions in the bunch

$$\alpha_u = \alpha_{u0},$$

$$\begin{aligned}
\alpha_p &= \alpha_{p0}[1 - (z - d/2)^2]/(d/2)^2, \\
\alpha_l &= \alpha_{l0}|z|/d, \\
-d &\leq z \leq 0.
\end{aligned} \tag{8}$$

Bunches with profiles (8), but with the same length and same total charge are compared. That means $|\alpha_{p0}| = (3/2)|\alpha_{u0}|$, $|\alpha_{l0}| = 2|\alpha_{u0}|$.

Typical dependence of the wake wave amplitude E_{mp} on bunch length for distributions (8), in the case of periodical field inside the uniform bunch (i.e. when $\alpha_1 < \alpha_{u0} < \gamma$) shown on Figure 4. For the uniform bunch this dependence is periodical (on Fig. 4 shown one period only). The wake wave amplitude behind the uniform bunch is maximum and equal to $E_{mp} = 2|\alpha_{u0}|\gamma/\Gamma^{1/2}$ [5], [6] when $d = (n + 1/2)\Lambda_b$ (n is integer) and equal to zero when $d = n\Lambda_b$. For the parabolic and linear profiles choice $d \approx (3/4)\Lambda_b$ is optimal. One can see also, that in the case $d > \Lambda_b$ the "parabolic" bunches becomes less effective and the wake wave amplitude behind the bunch with linear profile weakly depends on d . For a short bunch ($d < \Lambda_b/2$) $E_{mp}(d)$ almost not depends on the bunch profile.

4 Trains of Bunches

The results presented above demonstrates, that for the strong wake wave (with $E_{mp} \sim \tilde{E}$) excitation it is necessary relativistic either negatively charged bunch with $\alpha \leq -0.5$ or positively charged bunch with $\alpha \geq \gamma$. However, on experiments density of bunches is often much less of the plasma density, i.e. $|\alpha| \ll 1$. Therefore, it is naturally to try excite the strong waves by series of bunches. For the case of uniform bunches with the same densities this problem was investigated analytically in [7]. The wake wave amplitude behind the train of uniform bunches is maximum when $d_i = (n_i + 1/2)\Lambda_{bi}$ and $l_j = (n_j + 1/2)\Lambda_{pj}$, where d_i is the length of i -th bunch, l_j is the spacing between j -th and $(j + 1)$ -th bunches, Λ_{bi} is the wavelength in i -th bunch, Λ_{pj} is the wavelength behind j -th bunch, n_i and n_j are integer. The wake wave amplitude behind such optimized train

depending on number of bunch N is [7]:

$$E_{mp}(N) \approx (2/\beta)sh(|\alpha|\beta N), |\alpha| \ll 1. \quad (9)$$

From (9) follows, that the wave amplitude quickly grows as N increases and reaches the value \tilde{E} when

$$N \approx (|\alpha|\beta)^{-1} \ln[(\gamma + 1)^{1/2}/2 + (\gamma - 1)^{1/2}/2]. \quad (10)$$

The electric field in the optimised train, obtained by numerical simulation of equation (1) presented on Figure 5. The numerical results showed high exactness of the expressions (9) and (10).

Often in experiments using trains of bunches with $d_i = d = const, l_j = l = const$ and $|\alpha(z)| \ll 1$. Analytical discription of the fields excited in such trains is very difficult. The numerical simulation showed, that for profiles (8) in the case of d and l optimal for first bunch (see Sec. 3) the wake wave amplitude behind the train grows almost linearly up to $N = N_* \sim 0.5/|\alpha_{u0}| \cdot E_{mp}(N_*)$ increases as $|\alpha_{u0}|$ and (or) γ increases. When the wave becomes essentially nonlinear (when $N \geq N_*$) coherence of the waves exciting by the bunches breaking because of increasing of the wave length and the wave amplitude behind train decreases almost up to zero as N increases. Dependence $E_{mp}(N)$ for $N \gg N_*$ is nearly periodical.

It was ascertained, that trains of short bunches ($d \ll \Lambda_b/2$ and $l \sim \Lambda_p/2$) are not effective as in this case $E_{mp}(N) \leq E_{mp}(1)$. When $d \ll \Lambda_b$ and $l \ll \Lambda_p$ the train behaves as one bunch with averaged along the train length charge density.

5 Conclusions

The analytical and numerical results presented above throws light upon a number of questions of the theory of strong plasma waves excited by charged bunches. This results allow to choose parameters of the bunches optimal on future experiments.

The peculiarity of wake waves excitation by positively charged bunches are revealed. In particular, it was discovered, that the nonlinear wavelength in uniform positively

charged bunch decreases as charge density increases.

For the excitation of wake wave with $E_{mp} \sim \tilde{E}$ more suitable is a bunch, consisting of heavy particles (for example, protons), as light particles (electrons, positrons) lose considerable part of their energy on length comparable with the bunch length. This is easy to show by comparing of the bunch energy with the energy of wake wave.

Note, that the results presented in this paper are also suitable for the case of two-dimensional cylindrical bunches with radius a when $(\gamma c/\omega_p a)^2 \ll 1$ and $r^2 \ll a^2$ [9].

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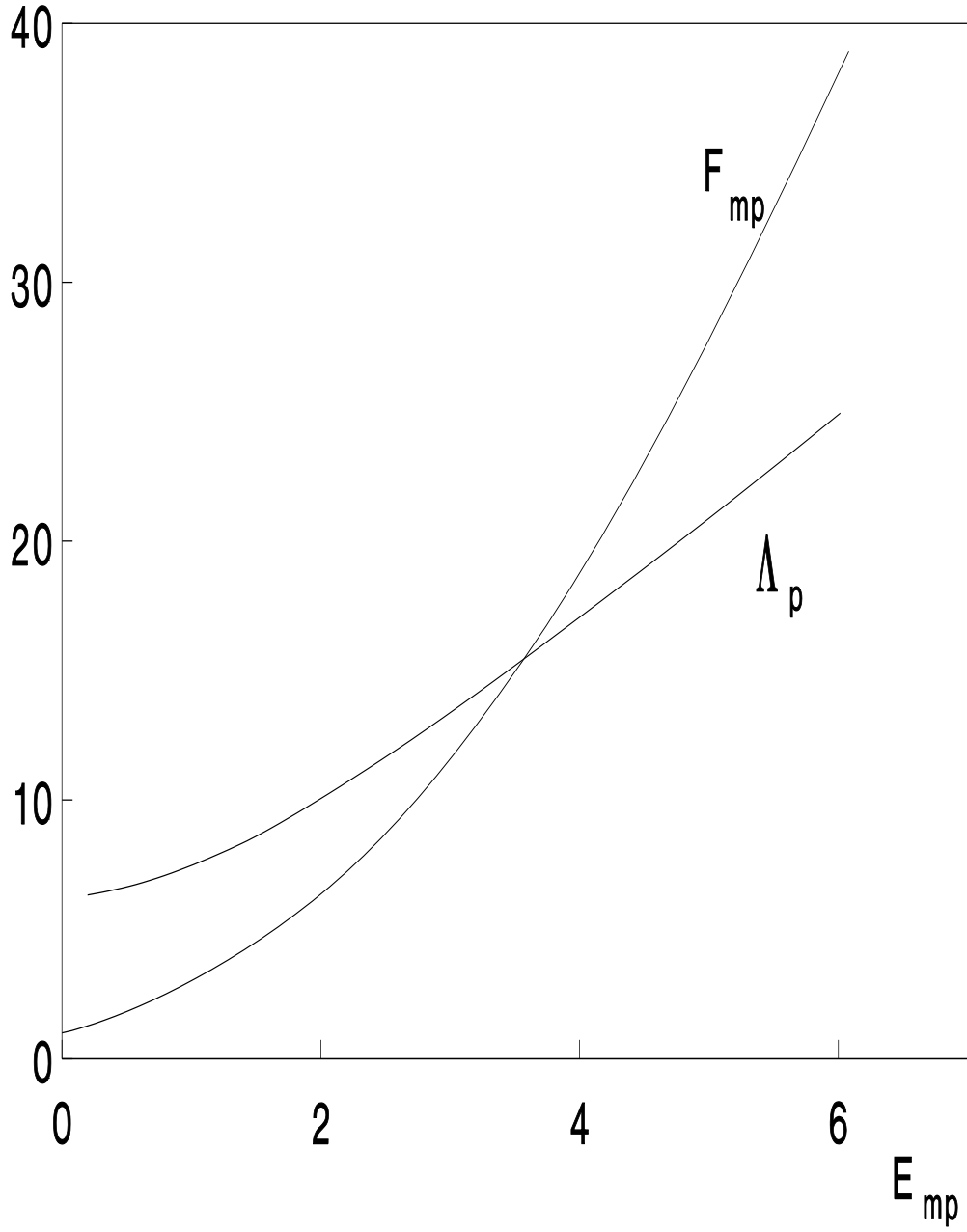


Figure 1. Dependence of the wake wavelength and maximum value of the electric potential on wake wave amplitude E_{mp} (in the dimensionless units); $\gamma \gg 1$.

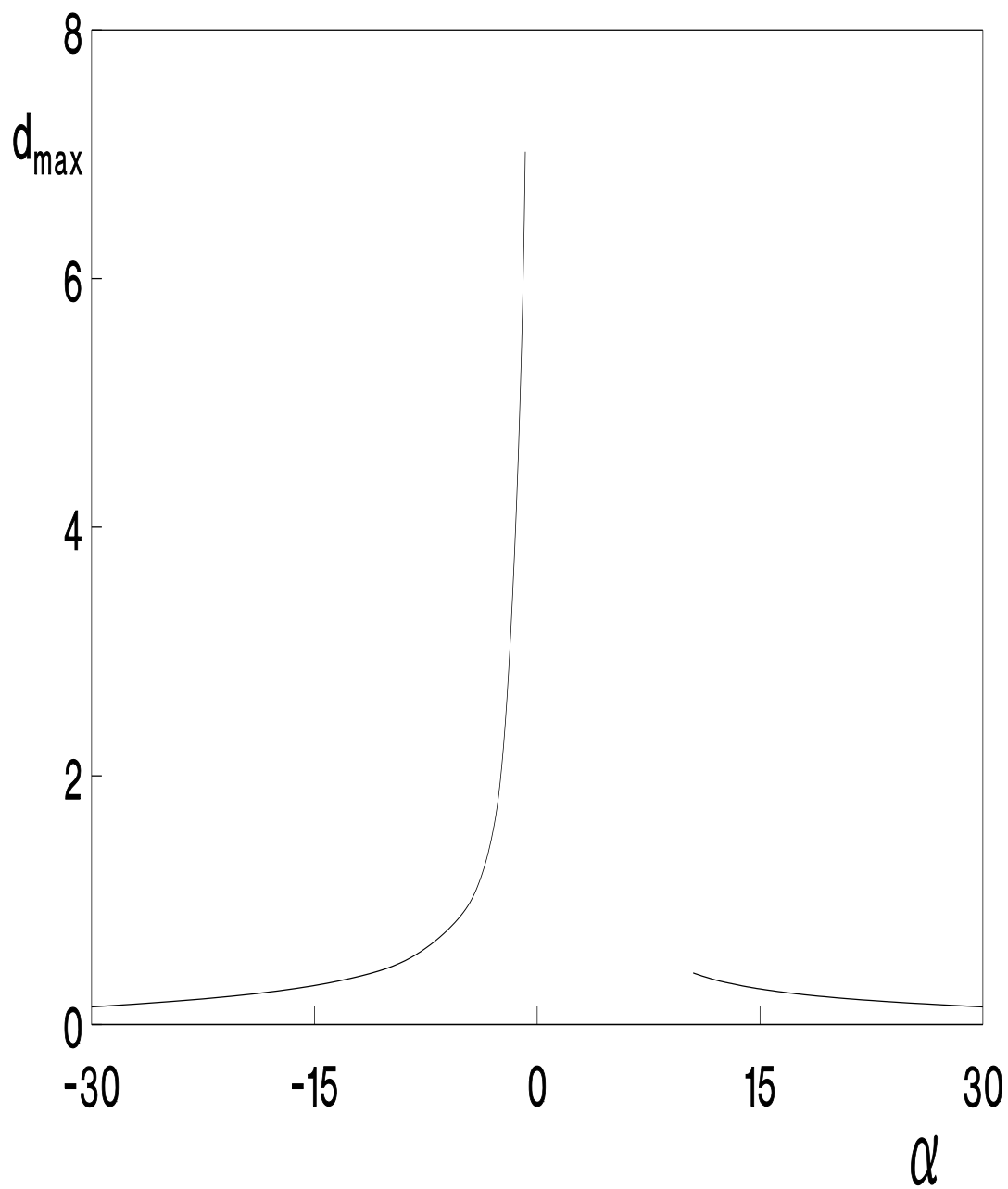


Figure 2. The maximum permissible bunch length d_{\max} depending on the bunch charge density ($\gamma = 10$).

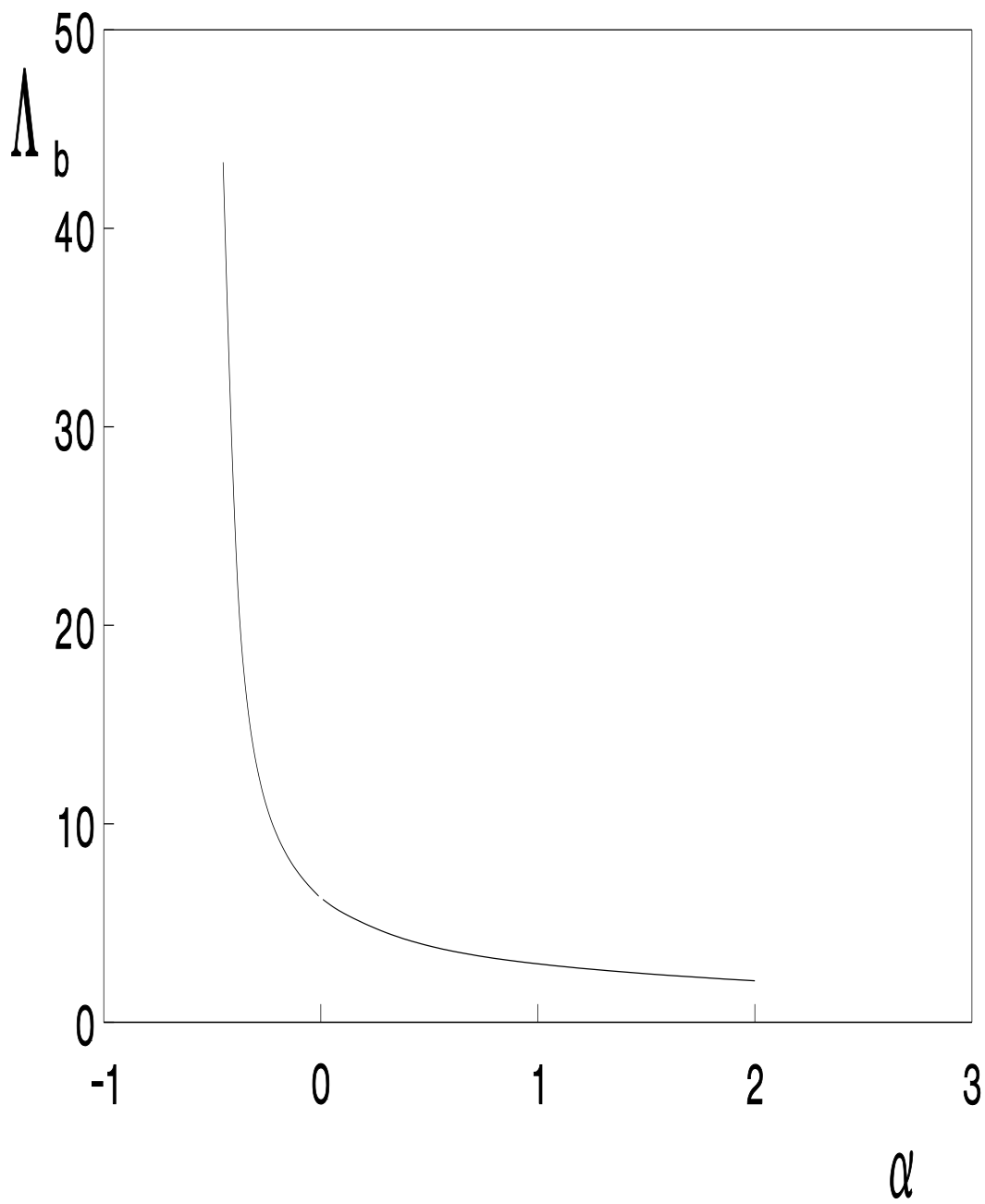


Figure 3. Dependence of the wavelength in uniform bunch on the charge density.

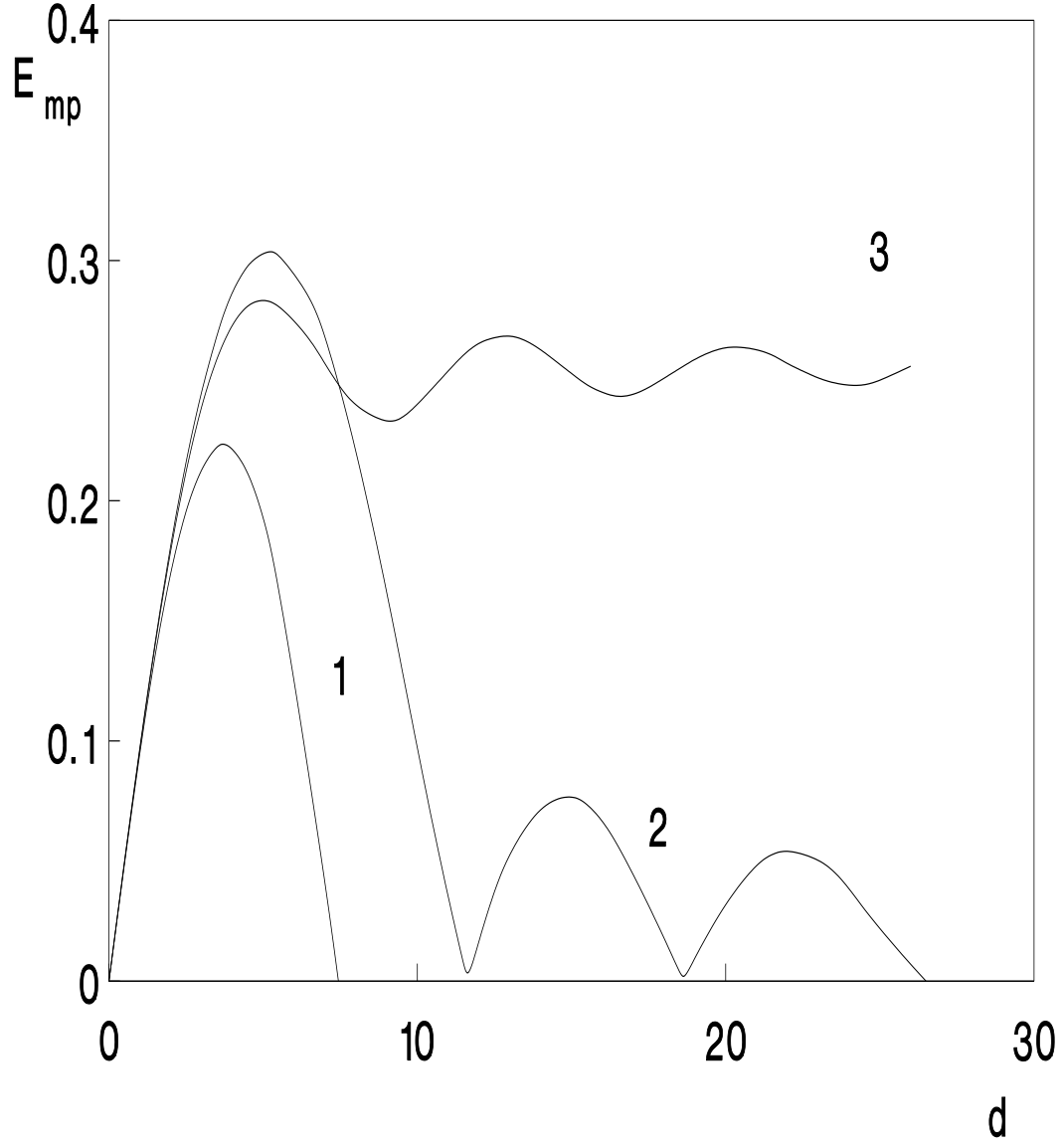


Figure 4. The wake wave amplitude behind bunch depending on the bunch length ($\alpha_{u0} = 0.1, \gamma = 10$). 1-uniform bunch, 2-parabolic profile of the bunch charge density, 3-linear profile case.

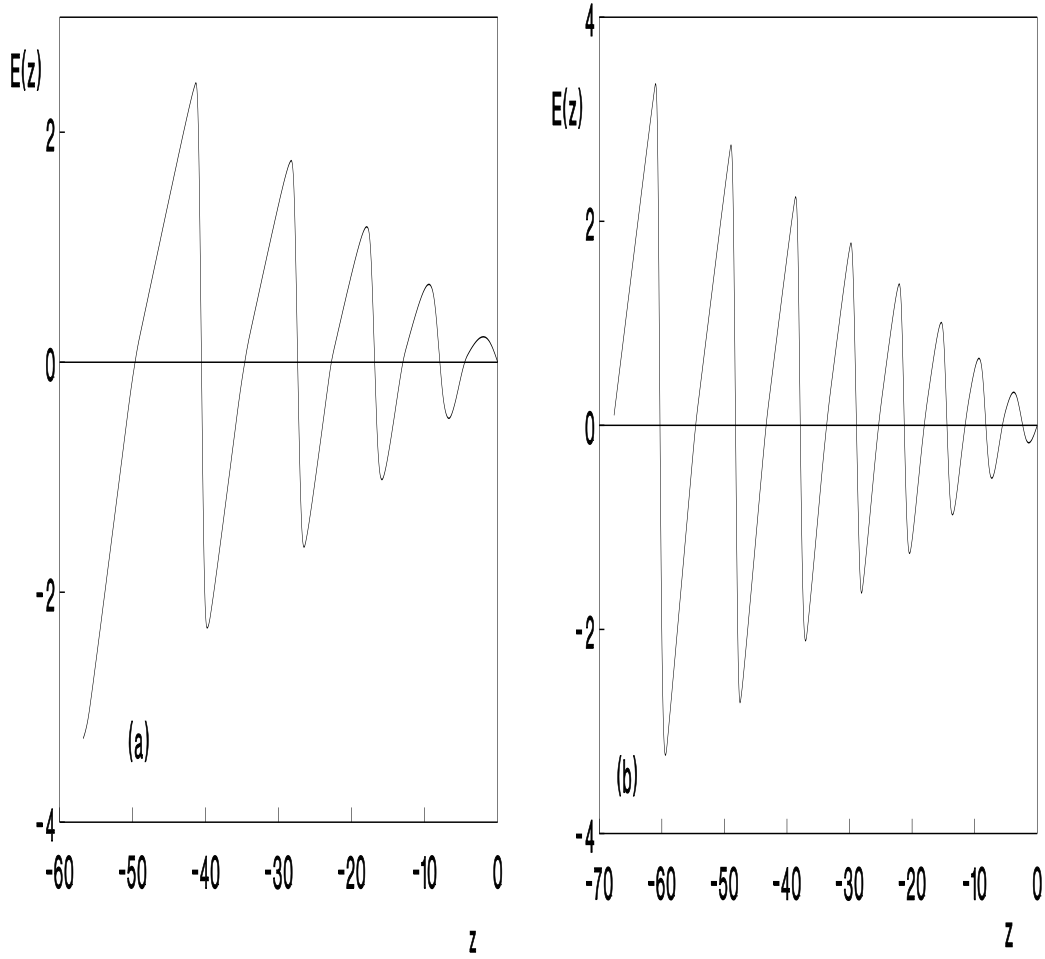


Figure 5. The electric field in optimized train of uniform bunches.

- a) The train of negatively charged bunches ($\alpha = -0.2, \gamma = 10$). Inside bunches $E > 0$. Negative E corresponds to space between bunches.
- b) The train of positively charged bunches ($\alpha = 0.2, \gamma = 10$). Inside bunches $E < 0$. Positive E corresponds to space between bunches.